

# Transfinite Version of Welter's Game

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## Abstract

We study the transfinite version of Welter's Game, a combinatorial game, which is played on the belt divided into squares with general ordinal numbers extended from natural numbers.

In particular, we obtain a straight-forward solution for the transfinite version based on those of the transfinite version of Nim and the original version of Welter's Game.

## 1 Introduction

### 1.1 Impartial game

This paper discusses only “impartial” games, by which we refer to games with the following characters:

- Two players alternately make a move.
- No chance elements (the effect of each move can be completely recognized before the move is made).
- Both players have complete knowledge of the game states.
- The game terminates in finitely many moves.
- Both players have the same set of options of moves in any position.

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In general, a game which satisfies the first three characters is called a combinatorial game.

Combinatorial games are called in normal (resp. *misère*) form if the player who makes the last move will win (resp. lose).

For the moment, we assume that there are only a finite number of positions that can be reached from the initial position, and a position may never be repeated (such a game is called short).

Let  $G$  be an impartial game position, we will express the symbol  $G \rightarrow G'$  means that  $G'$  can be reached by a single move from  $G$ .

The original version of Nim, Welter's Game and Green Hackenbush which will be discussed in chapter 1 and 2 are short games. In chapter 3, this assumption will be removed.

**Definition 1.1** (outcome classes). A game position is called an  $\mathcal{N}$ -position (resp. a  $\mathcal{P}$ -position) if the first player (resp. the second player) has a winning strategy.

Clearly, all impartial game positions are classified into  $\mathcal{N}$ -positions or  $\mathcal{P}$ -positions.

**Theorem 1.2** (Bouton[1]). If  $G$  is an  $\mathcal{N}$ -position, there exists a move from  $G$  to a  $\mathcal{P}$ -position. If  $G$  is a  $\mathcal{P}$ -position, there exists no move from  $G$  to a  $\mathcal{P}$ -position.

## 1.2 Nim

Nim is a well-known impartial game with the following rules:

- It is played with several heaps of tokens.
- The legal move is to remove any number of tokens (but necessarily at least one) from any heap.
- The end position is the state of no heaps of tokens.

Let us denote by  $\mathbb{N}_0$  the set of all nonnegative integers.

**Definition 1.3** (nim-sum). The value obtained by adding numbers in binary form without carry is called nim-sum. The nim-sum of nonnegative integers  $m_1, \dots, m_n$  is written as

$$m_1 \oplus \dots \oplus m_n.$$

The set  $\mathbb{N}_0$  is isomorphic to the direct sum of countably many  $\mathbb{Z}/2\mathbb{Z}$ 's. Also, the nim-sum operation can be extended naturally on  $\mathbb{Z}$  by using the 2's complement.

**Lemma 1.4** (Conway[4]). For integer  $n$ ,

$$n \oplus (-1) = -1 - n.$$

**Theorem 1.5** (Bouton[1]). We denote the Nim position with heaps of size  $m_1, \dots, m_n$  by  $(m_1, \dots, m_n)$ . Then,

$$\begin{aligned} m_1 \oplus \dots \oplus m_n \neq 0 &\iff \mathcal{N}\text{-position} \\ m_1 \oplus \dots \oplus m_n = 0 &\iff \mathcal{P}\text{-position.} \end{aligned}$$

### 1.3 Grundy number

Grundy number was introduced in attempt to develop the theory about general impartial games, which classifies positions of impartial games such as Nim.

**Definition 1.6** (minimum excluded number **mex**). Let  $T$  be a proper subset of  $\mathbb{N}_0$ . Then  $\text{mex } T$  is defined to be the least nonnegative integer not contained in  $T$ , namely

$$\text{mex } T = \min(\mathbb{N}_0 \setminus T).$$

**Definition 1.7** (Grundy number). We denote the end positions by  $E$ . Let  $G$  be a game position, and  $\{G'_1, \dots, G'_n\}$  the set of all positions reached by a single move from  $G$ . The value  $\mathcal{G}(G)$  is defined as follows:

$$\mathcal{G}(G) = \begin{cases} 0 & (G = E) \\ \text{mex}\{\mathcal{G}(G'_1), \dots, \mathcal{G}(G'_n)\} & (G \neq E). \end{cases}$$

Moreover,  $\mathcal{G}(G)$  is called the Grundy number of  $G$ .

**Theorem 1.8** (Sprague[2], Grundy[3]). We have the following for general short impartial games.

$$\begin{aligned} g(G) \neq 0 &\iff \mathcal{N}\text{-position} \\ g(G) = 0 &\iff \mathcal{P}\text{-position.} \end{aligned}$$

Thanks to this theorem, we only have to decide the Grundy number of a position for winning strategy in impartial games, and we can classify all game positions into  $\mathcal{N}$ -positions or  $\mathcal{P}$ -positions.

**Theorem 1.9** (Grundy[3]). Let  $(m_1, \dots, m_n)$  be a Nim position, we have

$$\mathcal{G}(G_1 + \dots + G_n) = m_1 \oplus \dots \oplus m_n,$$

where  $G = G_1 + \dots + G_n$ . Refer to [4] for the disjunctive sum  $+$  of game positions.

## 2 Welter's Game

### 2.1 Welter's Game

Welter's Game is an impartial game investigated by Welter in 1954. Since it was also investigated by Mikio Sato, it is often called Sato's game in Japan. The rules of Welter's Games are as follows:

0	1	•	•	4	•	6	•	8	9	...
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- It is played with several coins placed on a belt divided into several squares numbered with the nonnegative integers  $0, 1, 2, \dots$  from the left as shown in the figure above.
- The legal move is to move any one coin from its present square to any unoccupied square with a smaller number.
- The game terminates when a player is unable to move a coin, namely, the coins are jammed in squares with the smallest possible numbers as shown in the figure below.

•	•	•	•	4	5	6	7	8	9	...
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This game is equivalent to Nim with an additional rule that you cannot make two heaps with the same number of tokens.

Since this game is a short impartial game, our aim is to decide the Grundy number of a position.

### 2.2 Welter function

When an expression in what follows includes both nim-sum and the four basic operations of arithmetic without parentheses, we will make it a rule to calculate nim-sums prior to the others, and we express the nim-summation

by the symbol  $\sum^{\oplus}$ .

**Definition 2.1** (mating function). Mating function  $(x \mid y)$  is defined by

$$(x \mid y) = \begin{cases} 2^{n+1} - 1 & (x \equiv y \pmod{2^n}, \quad x \not\equiv y \pmod{2^{n+1}}) \\ -1 & (x = y). \end{cases}$$

Particularly, if  $x$  and  $y$  have different parities, then  $(x \mid y) = 1$ .

Then we have the following:

$(x \mid y) = (x - y) \oplus (x - y - 1)$ , and  $(x \mid y) = (x + a \mid y + a) = (x \oplus a \mid y \oplus a)$ .

**Lemma 2.2** (Conway[4]). For  $x, y, z \in \mathbb{Z}$ , we have the following equalities:

$$x \oplus (x \mid 0) = x - 1y \oplus (y \mid -1) = y + 1x \oplus y \oplus (x \mid y) = x \oplus y - 1.$$

**Lemma 2.3** (Conway[4]). For distinct integers  $x, y, z \in \mathbb{Z}$ , two of  $(x \mid y), (x \mid z), (y \mid z)$  are equal and the rest is greater than them

**Definition 2.4** (animating function). For any nonnegative integers  $a, b, c, d, \dots$ , a function  $f(x)$  of form

$$f(x) = (((x \oplus a) + b) \oplus c) + d \oplus \dots$$

is called an animating function.

If  $f$  and  $g$  are animating functions,  $f(g(x))$  and  $f^{-1}(x)$  are clearly animating functions. Also, we have  $f^{-1}(x) = (((\dots x \dots) - d) \oplus -b) \oplus a$ . Thus, the set of all animating functions forms a group with respect to composition.

**Theorem 2.5** (Conway[4]). Any animating function can be written as

$$f(x) = x \oplus n \oplus (x \mid p_1) \oplus \dots \oplus (x \mid p_k)$$

for some  $n, p_1, \dots, p_k \in \mathbb{Z}$ . Conversely, function the form of above is an animating function.

**Definition 2.6** (Welter function). Let  $(a_1, \dots, a_n)$  be a Welter's Game position. Then we define the value  $[a_1 \mid \dots \mid a_n]$  of Welter function at  $(a_1, \dots, a_n)$  as follows:

$$[a_1 \mid \dots \mid a_n] = a_1 \oplus \dots \oplus a_n \oplus \sum_{1 \leq i < j \leq n}^{\oplus} (a_i \mid a_j).$$

In the case of one coin, clearly  $[a_1] = a_1$ . In the case of two coins

$$[a_1 \mid a_2] = a_1 \oplus a_2 \oplus (a_1 \mid a_2) = a_1 \oplus a_2 - 1.$$

Let  $(a_1, \dots, a_n)$  be a position in Welter's Game and  $a_i, a_j$  the pair with the largest mating function value  $(a_i \mid a_j)$  (that is,  $a_i$  and  $a_j$  are congruent to each other modulo the highest possible power of 2 among all pairs).

By Lemma 2.3 mating function values  $(a_i | a_k)$  and  $(a_j | a_k)$  cancel each other for all other  $a_k$ 's. Thus, for example, if  $i = 1, j = 2$ , we can calculate the Welter function as follows:

$$\begin{aligned}
[a_1 | \cdots | a_n] &= a_1 \oplus \cdots \oplus a_n \oplus \sum_{1 \leq i < j \leq n}^{\oplus} (a_i | a_j) \\
&= a_1 \oplus a_2 \oplus (a_1 | a_2) \oplus \sum_{3 \leq k \leq n}^{\oplus} (a_1 | a_k) \\
&\quad \oplus \sum_{3 \leq k \leq n}^{\oplus} (a_2 | a_k) \oplus a_3 \oplus \cdots \oplus a_n \\
&\quad \oplus \sum_{3 \leq i < j \leq n}^{\oplus} (a_i | a_j) \\
&= a_1 \oplus a_2 \oplus (a_1 | a_2) \oplus a_3 \oplus \cdots \oplus a_n \\
&\quad \oplus \sum_{3 \leq i < j \leq n}^{\oplus} (a_i | a_j) \\
&= [a_1 | a_2] \oplus [a_3 | a_4 | \cdots].
\end{aligned}$$

This method is called splitting.

**Lemma 2.7** (Conway[4]).  $a_1 > a'_1, a_2 > a'_2, a_3 > a'_3, \dots$  are legal moves in Welter's Game, we have the following:

$$[a'_1 | a_2 | a_3 | \cdots] = [a_1 | a'_2 | a_3 | \cdots] \iff [a'_1 | a'_2 | a_3 | \cdots] = [a_1 | a_2 | a_3 | \cdots].$$

**Theorem 2.8** (Welter's Theorem[5]). The value of Welter function at each position in Welter's Game is equal to its Grundy number in Welter's Game. Namely, we have the following:

$$\mathcal{G}(a_1, \dots, a_n) = [a_1 | \dots | a_n].$$

### 2.2.1 Mating Method

Splitting reduces calculation of Welter function to that of the two argument Welter function and nim-sum.

When we mate pairs with the largest mating function value in order we have the following equality using splitting For Welter function of  $n$  arguments

$$[a_1 | \cdots | a_n] = \begin{cases} [a_1 | a_2] \oplus [a_3 | a_4] \oplus \cdots & (n : \text{even}) \\ [a_1 | a_2] \oplus [a_3 | a_4] \oplus \cdots \oplus [a_n] & (n : \text{odd}), \end{cases}$$

where  $(a_1 \mid a_2), (a_3 \mid a_4), \dots$  is arranged in order of the values of mating function.

By using this equality and formulas  $[a_1] = a_1$  and  $[a_1 \mid a_2] = a_1 \oplus a_2 - 1$ , we can easily compute the value of Welter function. This method is called Mating Method.

Welter function is an animating function with respect to each of its arguments. Since an animating function is a bijection on  $\mathcal{Z}$ , for any integers  $a_1, \dots, a_n$  and  $s$ , there exists a unique integer solution  $x$  for equation

$$[x \mid a_1 \mid \dots \mid a_n] = s.$$

Moreover if  $a_1, \dots, a_n$  are distinct nonnegative integers, and  $s$  is nonnegative,  $x$  is a nonnegative integer distinct from  $a_1, \dots, a_n$ .

### 3 Transfinite Game

#### 3.1 Transfinite Nim

First, we extend Nim into its transfinite version (Transfinite Nim) by allowing the size of the heaps of tokens to be a general ordinal number. The legal move is to replace an arbitrary ordinal number  $\alpha$  by a smaller number  $\beta$ . Therefore, Transfinite Nim may not necessarily be short.

Let us denote by  $\mathcal{ON}$  the class of all ordinal numbers. Later we see that the nim-sum operation can be extended naturally on  $\mathcal{ON}$ .

The following is known about general ordinal numbers.

**Theorem 3.1** (Cantor Normal Form theorem[6]). Every  $\alpha \in \mathcal{ON}(\alpha > 0)$  can be expressed as

$$\alpha = \omega^{\gamma_k} \cdot m_k + \dots + \omega^{\gamma_1} \cdot m_1 + \omega^{\gamma_0} \cdot m_0,$$

where  $k$  is a nonnegative integer,  $m_0, \dots, m_k \in \mathbb{N}_0 \setminus \{0\}$ , and  $\alpha \geq \gamma_k > \dots > \gamma_1 > \gamma_0 \geq 0$ .

Let  $\alpha_1, \dots, \alpha_n$  be ordinal numbers. Then, each  $\alpha_i$ ,  $i = 1, \dots, n$  is expressed by using finite by many common powers  $\gamma_0, \dots, \gamma_k$  as:

$$\alpha_i = \omega^{\gamma_k} \cdot m_{ik} + \dots + \omega^{\gamma_1} \cdot m_{i1} + \omega^{\gamma_0} \cdot m_{i0},$$

where  $m_{ik} \in \mathbb{N}_0$ .

Next, we will define the minimal excluded number of a set of ordinals and the Grundy number of a position in general Transfinite Game.

**Definition 3.2** (minimal excluded number **mex**). Let  $T$  be a proper subclass of  $\mathcal{ON}$ . Then  $\text{mex } T$  is defined to be the least ordinal number not contained in  $T$ , namely

$$\text{mex } T = \min(\mathcal{ON} \setminus T).$$

**Definition 3.3** (Grundy number). Let  $G$  be an impartial game (it may not necessarily be short). The value  $\mathcal{G}(G)$  is defined as

$$\mathcal{G}(G) = \text{mex}\{\mathcal{G}(G') \mid G \rightarrow G'\}.$$

**Theorem 3.4.** We have the following for Transfinite impartial games:

$$\begin{aligned} \mathcal{G}(G) \neq 0 &\iff \mathcal{N}\text{-position} \\ \mathcal{G}(G) = 0 &\iff \mathcal{P}\text{-position.} \end{aligned}$$

**Definition 3.5.** For ordinal numbers  $\alpha_1, \dots, \alpha_n \in \mathcal{ON}$ , we define their nim-sum as follows:

$$\alpha_1 \oplus \dots \oplus \alpha_n = \sum_k \omega^{\gamma_k} (m_{1k} \oplus \dots \oplus m_{nk}).$$

**Theorem 3.6.** For Transfinite Nim position  $(\alpha_1, \dots, \alpha_n) \subseteq \mathcal{ON}^n$  we have the following:

$$\mathcal{G}(\alpha_1, \dots, \alpha_n) = \alpha_1 \oplus \dots \oplus \alpha_n.$$

*Proof.* The proof is by induction. Let  $\alpha_1 \oplus \dots \oplus \alpha_n = \alpha$  ( $\alpha \in \mathcal{ON}$ ). We have to show that, for each  $\beta$  ( $\beta < \alpha$ ), there exists a position reached by a single move from  $(\alpha_1, \dots, \alpha_n)$  and that its Grundy number is  $\beta$ .

Let  $(\alpha_1, \dots, \alpha_n) \rightarrow (\beta_1, \dots, \beta_n)$ , by induction assumption we have

$$\mathcal{G}(\beta_1, \dots, \beta_n) = \beta_1 \oplus \dots \oplus \beta_n.$$

If  $\alpha = 0$ , no ordinal  $\beta$  ( $\beta < \alpha$ ) exists. We can assume  $\alpha > 0$ .

We can write  $\alpha$  and  $\beta$  as

$$\begin{aligned} \alpha &= \omega^{\gamma_k} \cdot a_k + \dots + \omega^{\gamma_1} \cdot a_1 + a_0 \\ \beta &= \omega^{\gamma_k} \cdot b_k + \dots + \omega^{\gamma_1} \cdot b_1 + b_0, \end{aligned}$$

where  $a_0, \dots, a_k, b_0, \dots, b_k \in \mathbb{N}_0$ . By definition,

$$a_s = m_{1s} \oplus \dots \oplus m_{ns}, \text{ for } s = 1, \dots, k.$$

Since  $\alpha > \beta$ , there exists  $s$  such that

$$a_s > b_s, a_t = b_t \text{ for all } t (< s).$$



As in the strategy of original Nim, since  $a_s > b_s$ , there is an index  $i$  such that

$$m_{is} > m_{is} \oplus a_s \oplus b_s.$$

We define

$$m'_{it} = m_{it} \oplus a_s \oplus b_s \text{ for all } t (\leq s)$$

and

$$\begin{aligned} \alpha'_i &= \omega^{\gamma_k} \cdot m_{ik} + \cdots \omega^{\gamma_s+1} \cdot m_{is+1} + \omega^{\gamma_s} \cdot m'_{is} \\ &\quad + \omega^{\gamma_s-1} \cdot m'_{is-1} + \cdots + \omega^{\gamma_0} \cdot m'_{i0}, \end{aligned}$$

where  $m_{is} \oplus a_s \oplus b_s = m'_{is}$ .

If we put  $\alpha'_i = \beta_i$ ,  $\alpha_j = \beta_j$  ( $j \neq i$ ). Then,  $\alpha_i > \beta_i$  and we have

$$\beta_1 \oplus \cdots \beta_{i-1} \oplus \beta_i \oplus \beta_{i+1} \oplus \cdots \beta_n = \beta$$

Therefore, for each  $\beta (< \alpha)$ , there is a position  $(\beta_1, \dots, \beta_n)$  reached by a single move from  $(\alpha_1, \dots, \alpha_n)$ .  $\square$

**Example 3.7.** In the case of position  $(1, \omega \cdot 2 + 4, \omega^2 \cdot 3 + 9, \omega^2 \cdot 2 + \omega \cdot 4 + 16, \omega^2 + \omega \cdot 5 + 25)$ :

Let us calculate the value of  $\alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 \oplus \alpha_5$ .

We get

$$\begin{aligned} \alpha_1 &= \omega^{\beta_2} \cdot m_{12} + \omega^{\beta_1} \cdot m_{11} + m_{10} = \omega^2 \cdot 0 + \omega \cdot 0 + 1 \\ \alpha_2 &= \omega^{\beta_2} \cdot m_{22} + \omega^{\beta_1} \cdot m_{21} + m_{20} = \omega^2 \cdot 0 + \omega \cdot 2 + 4 \\ \alpha_3 &= \omega^{\beta_2} \cdot m_{32} + \omega^{\beta_1} \cdot m_{31} + m_{30} = \omega^2 \cdot 3 + \omega \cdot 0 + 9 \\ \alpha_4 &= \omega^{\beta_2} \cdot m_{42} + \omega^{\beta_1} \cdot m_{41} + m_{40} = \omega^2 \cdot 2 + \omega \cdot 4 + 16 \\ \alpha_5 &= \omega^{\beta_2} \cdot m_{52} + \omega^{\beta_1} \cdot m_{51} + m_{50} = \omega^2 \cdot 1 + \omega \cdot 5 + 25. \end{aligned}$$

So, we have

$$\begin{aligned} m_{12} \oplus m_{22} \oplus m_{32} \oplus m_{42} \oplus m_{52} &= 0 \oplus 0 \oplus 3 \oplus 2 \oplus 1 \\ &= 0 \\ m_{11} \oplus m_{21} \oplus m_{31} \oplus m_{41} \oplus m_{51} &= 0 \oplus 2 \oplus 0 \oplus 4 \oplus 5 \\ &= 3 \\ m_{10} \oplus m_{20} \oplus m_{30} \oplus m_{40} \oplus m_{50} &= 1 \oplus 4 \oplus 9 \oplus 16 \oplus 25 \\ &= 5. \end{aligned}$$

Thus, by the definition of nim-sum in general ordinal number

$$\alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 \oplus \alpha_5 = \omega \cdot 3 + 5.$$

Therefore, this position is an  $\mathcal{N}$ -position, and the legal good move is  $\omega \cdot 2 + 4 \rightarrow \omega + 1$ .

### 3.2 Transfinite Welter's Game

In Transfinite version the size of the belt of Welter's Game is extended into general ordinal numbers, but played with finite number of coins. The legal move is to move one coin toward the left (jumping is allowed), and you cannot place two or more coins on the same square as in the original Welter's Game.

0	1	•	3	...	$\omega$	•	$\omega + 2$	...	$\omega^2$	...
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**Definition 3.8.** Let  $\alpha_1, \dots, \alpha_n \in \mathcal{ON}$ . Each  $\alpha_i$  can be expressed as  $\alpha_i = \omega \cdot \lambda_i + m_i$ , where  $\lambda_i \in \mathcal{ON}$  and  $m_i \in \mathbb{N}_0$ . Welter function in general ordinal numbers is defined as follows:

$$[\alpha_1 | \dots | \alpha_n] = \omega \cdot (\lambda_1 \oplus \dots \oplus \lambda_n) + \sum_{\lambda \in \mathcal{ON}}^{\oplus} [S_\lambda],$$

where  $[S_\lambda]$  is Welter function, and  $S_\lambda = \{m_n \mid \lambda_n = \lambda\}$ .

We obtain the following main theorem.

**Theorem 3.9.** Let  $\alpha_1, \dots, \alpha_n \in \mathcal{ON}$ . Grundy number of general position  $(\alpha_1, \dots, \alpha_n)$  in Transfinite Welter's Game is equal to its Welter function. Namely, we have the following:

$$\mathcal{G}(\alpha_1, \dots, \alpha_n) = [\alpha_1 | \dots | \alpha_n].$$

*Proof.* Let  $[\alpha_1 | \dots | \alpha_n] = \alpha$ . We have to show that, for each  $\beta (< \alpha)$ , there exists a position with Grundy number  $\beta$  which is reached by a single move from  $(\alpha_1, \dots, \alpha_n)$ .

Let  $(\alpha_1, \dots, \alpha_n) \rightarrow (\beta_1, \dots, \beta_n)$ . Then by the assumption of induction we have

$$\mathcal{G}(\beta_1, \dots, \beta_n) = [\beta_1 | \dots | \beta_n].$$

If  $\alpha = 0$ , there exist no  $\beta (< \alpha)$ . We can assume  $\alpha > 0$  and

$$\alpha = \omega \cdot \lambda + a_0 \text{ and } \beta = \omega \cdot \lambda' + b_0,$$

where  $\lambda, \lambda' \in \mathcal{ON}$ ,  $a_0, b_0 \in \mathbb{N}_0$ . Since  $\alpha > \beta$  we have

$$(\lambda > \lambda') \text{ or } (\lambda = \lambda' \text{ and } a_0 > b_0).$$

In the latter case, since  $a_0 = \sum_{\lambda \in \mathcal{ON}}^{\oplus} [S_\lambda] > b_0$ , from theory of Nim[1][2][3], there exists some  $\lambda_0$  and nonnegative integer  $c_0 (< [S_{\lambda_0}])$  such that

$$a_0 \oplus [S_{\lambda_0}] \oplus c_0 = b_0.$$

Next since  $[S_{\lambda_0}] > c_0$ , from theory of Welter function[4], there is an index  $i$  and  $m'_i (< m_i)$  such that  $m_i \in S_{\lambda_0}$  and  $[S'_{\lambda_0}] = c_0$ , where  $S'_{\lambda_0}$  is the set obtained from  $S_{\lambda_0}$  by replacing  $m_i$  with  $m'_i$ . Thus, the move from  $\alpha_i = \omega \cdot \lambda_i + m_i$  to  $\alpha'_i = \omega \cdot \lambda_i + m'_i$  changes its Grundy number from  $\alpha = \omega \cdot \lambda + a_0$  to  $\beta = \omega \cdot \lambda + b_0$ .

In the former case, as in Transfinite Nim,

there is an index  $i$  and  $\lambda'_i (< \lambda)$  such that  $(\lambda_1, \dots, \lambda_{i-1}, \lambda'_i, \lambda_{i+1}, \dots, \lambda_n)$  has Grundy number  $\lambda'$  and we can adjust the finite part of  $\alpha_i$  so that the resulting Welter function to be  $\beta$ .

Therefore, for each  $\beta (< \alpha)$ , there is a position reached by a single move from  $(\alpha_1, \dots, \alpha_n)$  and its Grundy number is  $\beta$ .  $\square$

**Corollary 3.10.** A position in Transfinite Welter's Game is a  $\mathcal{P}$ -position if and only if it satisfies the following conditions:

$$\begin{cases} \omega \cdot (\lambda_1 \oplus \dots \oplus \lambda_n) = 0 \\ \sum_{\lambda \in \mathcal{ON}}^{\oplus} [S_\lambda] = 0. \end{cases}$$

By this corollary, we can easily calculate a winning move in Transfinite Welter's Game by its Welter function.

**Example 3.11.** In the case of position  $(1, \omega \cdot 2 + 4, \omega \cdot 2 + 9, \omega^2 + \omega \cdot 4 + 16, \omega^2 + \omega \cdot 5 + 25)$ :

Let us calculate the value of  $[\alpha_1 | \alpha_2 | \alpha_3 | \alpha_4 | \alpha_5]$ . We get

$$\begin{aligned} \alpha_1 &= \omega^{\beta_2} \cdot m_{12} + \omega^{\beta_1} \cdot m_{11} + m_{10} = \omega^2 \cdot 0 + \omega \cdot 0 + 1 \\ \alpha_2 &= \omega^{\beta_2} \cdot m_{22} + \omega^{\beta_1} \cdot m_{21} + m_{20} = \omega^2 \cdot 0 + \omega \cdot 2 + 4 \\ \alpha_3 &= \omega^{\beta_2} \cdot m_{32} + \omega^{\beta_1} \cdot m_{31} + m_{30} = \omega^2 \cdot 0 + \omega \cdot 2 + 9 \\ \alpha_4 &= \omega^{\beta_2} \cdot m_{42} + \omega^{\beta_1} \cdot m_{41} + m_{40} = \omega^2 \cdot 1 + \omega \cdot 4 + 16 \\ \alpha_5 &= \omega^{\beta_2} \cdot m_{52} + \omega^{\beta_1} \cdot m_{51} + m_{50} = \omega^2 \cdot 1 + \omega \cdot 5 + 25. \end{aligned}$$

So, we have

$$\begin{aligned} m_{12} \oplus m_{22} \oplus m_{32} \oplus m_{42} \oplus m_{52} &= 0 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \\ &= 0 \\ m_{11} \oplus m_{21} \oplus m_{31} \oplus m_{41} \oplus m_{51} &= 0 \oplus 2 \oplus 2 \oplus 4 \oplus 5 \\ &= 1 \\ [m_{10}] \oplus [m_{20} | m_{30}] \oplus [m_{40}] \oplus [m_{50}] &= [1] \oplus [4 | 9] \oplus [16] \oplus [25] \\ &= 1 \oplus (4 \oplus 9 - 1) \oplus 16 \oplus 25 \\ &= 4. \end{aligned}$$

Therefore, by the definition Welter function for general ordinal number

$$[\alpha_1|\alpha_2|\alpha_3|\alpha_4|\alpha_5] = \omega + 4.$$

Since, this shows that we are in an  $\mathcal{N}$ -position, we will calculate a winning move.

First, we choose a move that satisfies the first condition of Corollary 3.10. Clearly we should not make a move that will change the coefficient of  $\omega^{\beta_2} = \omega^2$ . So we will choose a move that will change the coefficient of  $\omega^{\beta_1} = \omega^1$  to be 0. The same strategy in Transfinite Nim, shows that

$$(2 \oplus 2 \oplus 4 \oplus 5) \oplus 1 = 1 \oplus 1 = 0.$$

Thus, the only legal move is  $5 \rightarrow 5 \oplus 1 = 4$ . So, our good move is in  $\omega \cdot 5 + 25$ . Then in such moves we will search for a move that satisfy the second condition. It is obtained from the knowledge of Welter function.

The finite part should satisfy

$$1 \oplus [4 | 9] \oplus [x | 16] = 0.$$

So we have

$$x = 6.$$

Therefore, the only good move is  $\omega \cdot 5 + 25 \rightarrow \omega \cdot 4 + 6$ .

In fact

$$\begin{aligned} m_{12} \oplus m_{22} \oplus m_{32} \oplus m_{42} \oplus m_{52} &= 0 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \\ &= 0 \\ m_{11} \oplus m_{21} \oplus m_{31} \oplus m_{41} \oplus m_{51} &= 0 \oplus 2 \oplus 2 \oplus 4 \oplus 4 \\ &= 0 \\ [m_{10}] \oplus [m_{20} | m_{30}] \oplus [m_{40}] \oplus [m_{50}] &= [1] \oplus [4 | 9] \oplus [6 | 16] \\ &= 1 \oplus (4 \oplus 9 - 1) \oplus (6 \oplus 16 - 1) \\ &= 1 \oplus 12 \oplus 13 \\ &= 0. \end{aligned}$$

Thus, this position is a  $\mathcal{P}$ -position.

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